Calculus Review

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1 Fundamentals

1.1 Fundamental Theorem of Calculus

We begin with the most important corollary of the fundamental theorem of single-variable integral calculus. Let $f(x)$ be defined as the derivative of some function $F(x)$, i.e. $f(x) = \frac{d}{dx}F(x)$. Then, if f is "Riemann integrable" on $[a, b]$:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{1}$$

2 Multi-variable Calculus

2.1 Scalar Fields

A function $f(\vec{x})$ where $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ defines a **scalar field** (or *real-valued function*) if f associates a scalar value to every point in space. For example, $f(x, y) = x^2 + y^2$ defines a scalar field. When we set $z = f(x, y)$, we are defining a 3D surface by the scalar field.

2.2 General Form of a 3D Surface

We do not always want to set $z = f(x, y)$, rather we'd like to define an equation that constrains x, y, z such that $g(x, y, z) = 0$. Let us look at the equations for some common surfaces:

- $z^2 = x^2 + y^2$, a cone that grows negatively and positively out along the Z-axis from the origin. A cone that grows out along the x-axis is defined as $x^2 = y^2 + z^2$.
- $ax^2 + by^2 + cz^2 = 1$, an ellipsoid, where if $a = b = c$ then we have a *sphere*.
- $x^2 + y^2 = r^2$, a cylinder where its cross section is a circle of radius r.
- $z = x^2 + y^2$, a paraboloid (basically just a 3D parabola).
- $ax + by + cz = d$, a *plane*, also defined as $z = f(x, y) = Ax + By + D$.

2.3 Vector Functions

A vector function can be generally defined as a function that returns a vector. We can understand them as having two different types with different kinds of interpretations parametrized vector functions (which define *curves*) and vector fields which are more general.

2.3.1 Parametrized Vector-valued Functions

There are two main ways to represent vector-valued functions of some parameter vector $\vec{t}$, although in most cases for our purposes they will only be parametrized by a single variable t, or two variables p and q (or t and s). For both cases, $r(t)$ is a function that returns a vector, and $f(t), g(t), h(t)$ are *coordinate functions* of parameter t.

- ◇ Equation form: $r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$
- ◇ Vector form: $r(t) = \langle f(t), g(t), h(t)\rangle$

Differentiation. A single parameter allows for simple differentiation properties. For example, suppose $r(t)$ represents the position of a particle at time t, then we define the velocity vector $v(t)$ as follows:

- ◇ $v(t) = \frac{d}{dt}r(t) = \langle f'(t), g'(t), h'(t)\rangle$

Parametrization. We may want to know the *length* L of this curve $r(t)$ from $t = a$ to $t = b$. This is actually defined as:

$$L = \int_a^b \left|\left|\frac{d}{dt}r(t)\right|\right|dt \tag{2}$$

We may also want to *reparametrize* the function based upon a new parameter s, where t is a function of s ($t(s)$), such that we know where we are after travelling a distance d along the curve. I.e., we would like to be able to reparametrize the function $r(t)$ in terms of a new parametrization of $t \to t(s)$ such that $r(t(s) = d)$ gives us the point we arrive at. This is desirable, and is obtained by This defines the arclength function of the curve $s(t)$. In general we have, and can then easily solve for $t(s)$ after computing this:

$$s(t) = \int_0^t \left|\left|\frac{d}{dt}r(t)\right|\right|dt \tag{3}$$

Example. Let $r(t) = \langle sin(2t), cos(2t), 4t\rangle$, and suppose we have travelled a distance $\frac{5}{4}\pi$ starting from $t = 0$, what point are we at now?

- ◇ $r'(t) = \frac{d}{dt}r(t) = \langle 2cos(2t), -2sin(2t), 4\rangle$
- ◇ $||r'(t)|| = \sqrt{4cos^2(2t) + 4sin^2(2t) + 16} = \sqrt{20} = 2\sqrt{5}$
- ◇ $s(t) = \int_0^t 2\sqrt{5}dt = 2\sqrt{5}t$
- ◇ So, $t(s) = \frac{1}{2\sqrt{5}}s = \frac{\sqrt{5}}{10}s$
- ◇ So, $r(t(s)) = \langle sin(\frac{\sqrt{5}}{5}s), cos(\frac{\sqrt{5}}{5}s), \frac{2\sqrt{5}}{5}s\rangle$
- ◇ Finally giving us the property that $r(t(s = d))$ for any distance d gives us the point we arrive at after distance d from $t = 0$, thus $r(t(\frac{5}{4}\pi)) = \langle sin(\frac{\sqrt{5}}{4}\pi), cos(\frac{\sqrt{5}}{4}\pi), \frac{\sqrt{5}}{2}\pi\rangle$.

3 Vector Field

Like a scalar field, a vector field assigns a vector to every point in space. Vector fields are different from what is described above because they generally do not define curves or surfaces, but most generally, is a function that defines a tangent vector to each point on a "differentiable manifold". In specific terms, we can use vector fields to describe force, the velocity of a moving flow in space, etc. We give vector fields a full section because of their many many properties and qualities.

3.1 Formal Definition

We define a vector field in $\mathbb{R}^3$, although this can easily be generalized to $\mathbb{R}^n$ or particularized to $\mathbb{R}^2$. Let $\mathbb{E}$ be a subset of $\mathbb{R}^3$ a **vector field** on $\mathbb{E}$ is a function $\mathbf{F}$ that assigns to each point $(x, y, z) \in \mathbb{E}$ a 3D vector $\mathbf{F}(x, y, z)$. We can represent $\mathbf{F}$ via the following equation, expressed by a set of scalar fields P, Q, R which are called the *component functions* of $\mathbf{F}$:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} \tag{4}$$

3.2 Gradient

The gradient of a (scalar field) function $f(\vec{x})$ is defined as the vector of partial derivatives of f with respect to each variable $x_i \in \vec{x}$:

$$\nabla f = \langle \frac{df}{dx_1}, \frac{df}{dx_2}, \dots \frac{df}{dx_n} \rangle \tag{5}$$

It turns out that the gradient of a function not only defines a vector field, but defines a specific kind of vector field, a gradient field, or a **conservative** field, which we describe below. Note that we have the following property (expressed in $\mathbb{R}^3$) which allows us to relate the gradient field to the general form of a vector field, having $P = f_x, Q = f_y, R = f_z$, (let $(x, y, z) = \vec{x}$):

$$\mathbf{F}(\vec{x}) = \nabla f(\vec{x}) = \langle f_x(\vec{x}), f_y(\vec{x}), f_z(\vec{x}) \rangle = \langle P(\vec{x}), Q(\vec{x}), R(\vec{x}) \rangle \tag{6}$$

3.3 Conservative Vector Field

A conservative vector field is simply just the gradient of a scalar field. However, when a vector field is not conservative this means that it is not the gradient of a scalar field, which in loose interpretation, is like a function that cannot be integrated. The important characteristics of a conservative vector field are understood in the context of **line integrals**, explained below. However, one fact is that $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field if and only if (for all intents and purposes here):

$$\frac{dP}{dy} = \frac{dQ}{dx} \tag{7}$$

4 Line Integrals

Rather than integrating over an interval $[a, b]$, we would like to know what happens when we integrate over a *curve* C. Let us begin in $\mathbb{R}^2$ by defining this plane curve C, which is really just a t-parametrized vector-valued function $C(t) = \langle x(t), y(t)$, or $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, such that $a \leq t \leq b$ gives a closed parametric constraint on t. This is a particularization of what is described in Section 2.1, although we add one particular quality to C, namely, that it is **smooth**; i.e., $r'(t)$ is continuous and $r'(t) \neq \langle 0, 0 \rangle \; \forall t$.

Similarly to a Riemann sum or Riemann integral, we can define a line integral of a function f of two variables over C as follows. We enumerate this understanding in bullet points to provide an intuition that is as clear as possible for what a line integral is doing (or rather, what we would like it to do, since we haven't defined it yet):

- Let us divide the parameter interval $[a, b]$ into n (infinitesimally) small subintervals $[t_{i-1}, t_i]$; let $x_i = x(t_i), y_i = y(t_i)$
- We now have points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ that divide C into n sub-arcs s_i with lengths $\Delta s_1, \dots, \Delta s_i, \dots, \Delta s_n$.
- For example, s_1 is the sub-arc between points (x_0, y_0) and (x_1, y_1), and Δs_1 is its length (see Equation 2); fundamentally this corresponds to the parametrization interval $[t_0, t_1]$.
- Now consider a point *on the sub-arc* s_i; this point is parametrized by some t_i^* such that $t_{i-1} \le t_i^* \le t_i$; this point is thus expressed as (x_i^*, y_i^*).
- We can now define the **line integral** of f along C via infinitesimal calculus according to the arc-length differential ds of C, which is really just the length of an infinitesimally small sub-arc s.

$$\int_C f(x, y) ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i \tag{8}$$

However, as we learned from Section 2.3.1, we know that we can directly compute an arclength (even a differential one) ds via Equation 2. With this, we can express the direct computation of a line integral in terms of its parametrization (which does not matter, so long as C travels along the parametrization from a to b exactly once):

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \left|\left| \frac{d}{dt} f(x(t), y(t)) \right|\right| dt \tag{9}$$

4.1 Line Integral of a Piecewise-smooth Curve

A **piecewise-smooth curve** C is defined as a union of a finite number of smooth curves $C_1, C_2, \dots, C_n$ such that the terminal point of C_i is the initial point of C_{i+1}; i.e., all of the curves are connected. We have a straightforward definition of the line integral of f over C as the sum of all of the line integrals of f over C_i:

$$\int_C f(x, y) ds = \sum_{i=1}^{n} \int_{C_i} f(x, y) ds \tag{10}$$

4.2 Line Integrals with Respect to a Variable

In Equation 8 we must clarify that we are taking the line integral with respect to the arclength of C. However, we can also take the line integral with respect to other variables, such as x and y; note the necessity of being in parametric form:

$$\begin{aligned} \int_C f(x, y) dx &= \int_a^b f(x(t), y(t)) x'(t) dt \\ \int_C f(x, y) dy &= \int_a^b f(x(t), y(t)) y'(t) dt \end{aligned} \tag{11}$$

4.3 Line Integrals on Vector Fields

We now consider our vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ as a continuous **force field** on $\mathbb{R}^3$. With similar reasoning as in the beginning of Section 4, we find that the **work** W done by a $\mathbf{F}$ along a curve C is the following, where $\mathbf{T}(x, y, z)$ is the unit tangent vector at point (x, y, z):

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) ds \tag{12}$$

This says that work is the line integral with respect to the arclength of the tangential component of force. If C is given by $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and $a \leq t \leq b$, then $\mathbf{T}(t) = r'(t)/||r'(t)||$, and we can derive many equivalent forms of the equation:

$$W = \int_C \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C P dx + Q dy + R dz \tag{13}$$

5 Fundamental Theorems of Line Integrals

We can generalize the fundamental theorem of calculus (see Section 1) to be just a particular instance of the fundamental theorem of line integrals. Let C be a smooth curve given by a vector function $r(t)$ and constrained by $a \leq t \leq b$; let f be a differentiable function whose gradient vector ∇f is continuous on C, then:

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)) \tag{14}$$

This means that we can evaluate the line integral *of a conservative vector field* by simply knowing the value of f at the endpoints of C.

5.1 Green's Theorem

Let $\oint_C$ denote the line integral of a **closed** curve C that moves along in *positive-orientation* (i.e., it defines a domain which is always to the left by moving along C counter-clockwise) given by vector function $r(t)$ and $a \leq t \leq b$. Let C also be a piecewise-smooth, simple closed curve that defines a region D which it bounds. Also, P and Q must have continuous partial derivatives on D. Then Green says that:

$$\oint_C P dx + Q dy = \iint_D \Big(\frac{dQ}{dx} - \frac{dP}{dy} \Big) dA \tag{15}$$

5.2 Curl

Curl is a vector operation on vector field $\mathbf{F} = \langle P, Q, R \rangle$. Let us represent it in $\mathbb{R}^3$. First, we must consider the gradient operator ∇ as a vector, after which the definition of curl extends naturally:

$$\nabla = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \Big(\frac{dR}{dy} - \frac{dQ}{dz} \Big) \mathbf{i} + \Big(\frac{dP}{dz} - \frac{dR}{dz} \Big) \mathbf{j} + \Big(\frac{dQ}{dx} - \frac{dP}{dy} \Big) \mathbf{k} \tag{16}$$

Note the following interesting theorem, the curl of any conservative vector field is zero:

$$\operatorname{curl} \nabla f = \mathbf{0} \tag{17}$$

This gives us another way to determine if a vector field is conservative — take its curl and if it does not equal zero, then the vector field is not conservative.

5.3 Divergence

A very simple operation on a vector field $\mathbf{F} = \langle P, Q, R \rangle$, expressed as $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$.

$$\operatorname{div} \mathbf{F} = \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \tag{18}$$

We find another interesting theorem, the divergence of the curl of a vector field is zero. This can be easily proven by working out $\nabla \cdot (\nabla \times F)$.

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0 \tag{19}$$

Laplace Operator. The Laplace operator ∇^2 occurs naturally in the context of divergence, it is simply the sum of the second-order partial derivatives; i.e., $\nabla^2 = \nabla \cdot \nabla$:

$$\nabla^2 \mathbf{F} = \nabla \cdot (\nabla \mathbf{F}) = \frac{d^2 P}{dx^2} + \frac{d^2 Q}{dy^2} + \frac{d^2 R}{dz^2} \tag{20}$$

5.4 Vector form of Green's Theorem

It can be derived that, if we consider a line integral of $\mathbf{F}$ with regards to the unit normal vector $n(t)$, we obtain the following useful formulation, which means that the line integral of the normal component of $\mathbf{F}$ along the curve is equal to the double integral of the divergence of $\mathbf{F}$ over the region D enclosed by C:

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA \tag{21}$$

6 Surfaces

6.1 Parametric Surfaces

We define a parametric surface with a two-parameter (u, v) vector-valued function:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

A surface is obtained from this as (u, v) ranges across an entire domain D. We can now obtain the area of this surface (surface area) of the surface S defined by $\mathbf{r}(u, v)$ and D:

$$\text{Area}(S) = \iint_D \left|\left| \left(\frac{d}{du} \mathbf{r} \right) \times \left(\frac{d}{dv} \mathbf{r} \right) \right|\right| \, dA \tag{22}$$

We see that are integrating over the norm of the partial derivative of $\mathbf{r}$ with respect to u crossed with $\mathbf{r}$'s partial derivative with respect to v.

Unit normal vector $\mathbf{n}$ of a surface S defined by $\mathbf{r}(u, v)$ is:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||} \tag{23}$$

We can now obtain several equivalent expressions for the surface integral of a vector field, also called its **flux**:

$$\iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \tag{24}$$

6.2 Stokes' Theorem

A higher-dimensional version of Green's theorem, more general. Let S (defined by $\mathbf{r}(u, v)$) be a smooth surface bounded by a smooth closed curve C (defined by $\mathbf{r(t)}$), and let $\mathbf{F}$ be a vector field defined continuously on S. Then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot dS \tag{25}$$

6.3 Divergence (Gauss's) Theorem

Let E be a simple solid region and let S be the boundary surface of E. Let $\mathbf{F}$ be a vector field. Then we have the following, which means that the flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E.

$$\iint_S \mathbf{F} \cdot dS = \iiint_E \operatorname{div} \mathbf{F}\, dV \tag{26}$$